# 1 Triangles: Basics

This section will cover all the basic properties you need to know about triangles and the important points of a triangle. **You should know all of this by heart!** This is especially true when we cover more advanced topics in geometry later on because I will not be spending time in the future to cover basic material.

Let $ABC$ be a (non-degenerate) triangle. Let $G$, $I$, $O$, $H$ be the centroid, incentre, circumcentre and orthocentre of the triangle respectively. Let $I_A, I_B, I_C$ be the excentres of the excircle opposite $A, B, C$ respectively. Let $r, R$ be the inradius and circumradius of triangle $ABC$. Let $K$ be the area of $ABC$.

Let us very briefly review these points, the proof of their existence and their properties.

<table>
<thead>
<tr>
<th>Centroids:</th>
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<tbody>
<tr>
<td>Let $X, Y, Z$ be the midpoints of $BC, CA, AB$ respectively. Please solve the following basic problems.</td>
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<tr>
<td>1. $AX, BY, CZ$ intersect at a point $G$. This point is called the centroid of $ABC$.</td>
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<td>2. Prove that $\frac{AG}{GX} = \frac{BG}{GY} = \frac{CG}{GZ} = 2$.</td>
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<td>3. Let $G'$ be the reflection of $G$ across $X$. Prove that $BGCG'$ is a parallelogram.</td>
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<th>Circumcentre:</th>
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<tr>
<td>Let $X, Y, Z$ be the midpoints of $BC, CA, AB$ respectively.</td>
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<tr>
<td>1. The three lines perpendicular to $BC, CA, AB$ passing through $X, Y, Z$ respectively, are concurrent at a point $O$ and $O$ is equidistant to $A, B, C$. This point is called the circumcentre of $\triangle ABC$, whose radius $R$ is called the circumradius of $\triangle ABC$.</td>
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<td>2. Prove that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.</td>
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<td>3. Prove that $K = \frac{abc}{4R}$.</td>
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Orthocentre
Let \( D, E, F \) be the foot of the perpendicular from \( A, B, C \) on \( BC, CA, AB \) respectively. Please prove the following facts.

1. By noting that \( XYZ \) is similar to \( ABC \), prove that \( AD, BE, CF \) are concurrent. This point is called the orthocentre of \( \triangle ABC \).

2. From (1), conclude that
   \[
   \frac{AH}{OX} = \frac{BH}{OY} = \frac{CH}{OZ} = 2.
   \]

3. From the section on centroids, conclude that \( H, G, O \) are collinear. This line is called the Euler Line of triangle \( \triangle ABC \).

4. Let \( H_A, H_B, H_C \) be the midpoints of \( AH, BH, CH \) respectively. Let \( O' \) be the midpoint of \( HO \). Prove that \( O' \) is equidistant to \( D, E, F, X, Y, Z, H_A, H_B, H_C \). Conclude that these nine points are concyclic. The circle passing through these nine points is called (appropriately) the nine-point circle of \( \triangle ABC \).

5. Amongst the nine points on the nine-point circle, find the pairs of points which form a diameter of this circle. (This should give you many angles which are equal to \( 90^\circ \)).

6. Prove that amongst the points of \( A, B, C, H \), the orthocentre of any three of these points is the fourth point.

7. Prove that the point which is the image of reflection of \( H \) across any side of the triangle, is on the circumcircle of the triangle.
## Incentres:

1. The internal angle bisectors of angle $A, B, C$ intersect at a point $I$. This point is called the **incentre** of $\Delta ABC$.

2. The incentre $I$ is the centre of a circle which is tangent to the segments $BC, CA, AB$, say at $P, Q, R$ respectively. The **inradius** $r$ is the radius of this circle. Express $r$ in terms of the side lengths of the triangle $a, b, c$ and its area $K$.

3. Let $s = (a + b + c)/2$ be the semi-perimeter of $\Delta ABC$. Prove that $|AQ| = |AR| = s − a$. $|BR| = |BP| = s − b$ and $|CP| = |CQ| = s − c$.

4. Let the internal angle bisector of $A$ intersect $BC$ at $T$. Prove that

$$\frac{|AB|}{|AC|} = \frac{|BT|}{|TC|}.$$

5. Let the internal angle bisector of angle $A$ of $\Delta ABC$ intersect the circumcircle of $ABC$ at $M$. Prove that $|MB| = |MC| = |MI|$. i.e. $M$ is on the midpoint of the arc $BC$ not containing $A$, on the circumcircle of $\Delta ABC$ and the circumcircle of $\Delta BIC$ has centre $M$.

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*A common mistake is to think that $AP, BQ, CR$ are the angle bisectors of $A, B, C$. This is not true, and in fact is never true for scalene triangles!*
**Excentres:**

1. The internal angle bisector of $A$, and the external angle bisectors $B, C$ intersect at a point $I_A$. This is called the **excentre opposite** $A$. Analogous definition follows for the excentre opposite $B$ and $C$. The circle with centre $I_A$ tangent to $BC, AB, AC$ is called the **excircle opposite** $A$.

2. Prove that the excircle opposite $A$ touches $BC$ at a point $P_A$ which is the reflection of $P$ across the midpoint of $BC$. Conclude that $AB + BP_A = P_AC + CA$, meaning $P_A$ splits the broken line $AB, BC, CA$ in half.

3. From (2), prove that the excircle opposite $A$ touches ray $AB, AC$ at points whose distance from $A$ is the semi-perimeter of $\Delta ABC$.

4. Let $P'$ be the point on the incircle of $\Delta ABC$ such that $P'P$ is a diameter of the incircle. Prove that $A, P', P_A$ are collinear.

5. Prove that $I_A, C, I_B$ are collinear. Similarly, $I_B, A, I_C$ are collinear and $I_C, B, I_A$ are collinear.

6. Prove that $I_A P_A, I_B P_B, I_C P_C$ are concurrent. Prove that this point of concurrency is the circumcentre of $\Delta I_AI_BI_C$.

7. Prove that $I$ is the orthocentre of $\Delta I_AI_BI_C$.

8. Let the external angle bisector of $A$ intersect line $BC$ at $T'$. Prove that

$$\frac{|AB|}{|AC|} = \frac{|BT'|}{|TC'|}$$

Find an interpretation of this equation if this external angle bisector is parallel to $BC$. Compare this also for the analogous result for internal angle bisectors.
Exercises For Tuesday, September 23, 2008:

1. Given triangle $ABC$ such that $\angle A = 60^\circ$, with orthocentre $H$, incentre $I$ and circumcentre $O$. Prove that $B, C, H, I, O$ are concyclic. In fact, if a triangle has the property such that $B, C$ and two of these points are concyclic, then $\angle A = 60^\circ$.

(This problem can be called, why problem proposers love setting an angle to be $60^\circ$.)

2. Let $ABCD$ be a convex quadrilateral (with vertices appearing in that order) such that $\angle DAC = 80^\circ$, $\angle ACD = 50^\circ$, $\angle BDC = 30^\circ$ and $\angle DBC = 40^\circ$. Prove that $\triangle ABC$ is equilateral.

3. Given a triangle $ABC$ with $\angle A = 60^\circ$, let $D$ be any point on side $BC$. Let $O_1$ be the circumcentre of $ABD$ and $O_2$ be the circumcentre of $ACD$. Let $M$ be the intersection of $BO_1$ and $CO_2$ and $N$ be the circumcentre of $DO_1O_2$. Prove that $MN$ passes through a point independent of $D$.

4. Given triangle $ABC$, let $D$ be the foot of the perpendicular from $A$ on $BC$ and $M$ be the midpoints of $BC$. Points $P, Q$ are on rays $AB$ and $AC$ respectively such that $|AP| = |AQ|$ and $M$ is on line $PQ$. Let $S$ be the circumcentre of $APQ$. Prove that $|SD| = |SM|$.

5. Given an acute-angled triangle $ABC$, let $H$ be the orthocentre of $ABC$, $K$ be the midpoint of $AH$ and $M$ be the midpoint of $BC$. Prove that the intersection of the angle bisectors of $\angle ABH$ and the angle bisector of $\angle ACH$ lies on the line $KM$. 

Exercises for Tuesday, September, 30, 2008:

1. Let $ABC$ be a triangle with $|AC| > |AB|$. Let the $X$ be the intersection of the perpendicular bisector of $BC$ and the internal angle bisector of $A$. Let $P, Q$ be the foot of the perpendicular from $X$ on $AB$ extended and $AC$. Let $Z$ be the intersection of $PQ$ and $BC$. Find the ratio $BZ/ZC$.

2. Given a triangle $ABC$ with orthocentre $H$, centre of the nine-point circle $O$ and altitude $AD$, let $P$ be the midpoint of $AH$ and $Q$ be the midpoint of $PD$. Prove that $OQ$ is parallel to $BC$.

3. Given an acute-angled triangle $ABC$, let $H$ be the orthocentre of $ABC$, $K$ be the midpoint of $AH$ and $M$ be the midpoint of $BC$. Prove that the intersection of the internal angle bisector of $\angle ABH$ and the internal angle bisector of $\angle ACH$ lies on the line $KM$. (From last week)

4. Let $ABC$ be an acute-angled triangle with $|AB| < |AC|$, altitudes $AD, BE, CF$ and orthocentre $H$. Let $P$ be the intersection of $BC$ and $EF$, $M$ be the midpoint of $BC$ and $Q$ be the intersection of the circumcircle of $MBF$ and $MCE$.

(a) Prove that $\angle PQM = 90^\circ$.

(b) Conclude that $P, H, Q$ are collinear.

(c) Let $\omega$ be the circle passing through $B, C, E, F$. What are the images each of the points $A, B, C, D, E, F, P, Q, M$, the midpoints of $AB, BC, CA$ and the midpoints of $AH, BH, CH$ under the inversion about $\omega$?

5. A quadrilateral is said to be bicentric if it contains a circumcircle (i.e. $ABCD$ is cyclic) and an incircle. Let $a, b, c, d$ be the side lengths of a bicentric quadrilateral $ABCD$, with the lengths appearing in that order around the quadrilateral. Let $s$ be the semiperimeter of the quadrilateral.

(a) Prove that $a + c = b + d$. (This is in fact a necessary and sufficient condition for a quadrilateral to have an incircle.)

(b) Let $r$ be the radius of the incircle of $ABCD$ and $R$ be the radius of the circumcircle of $ABCD$. Prove that

$$r = \frac{\sqrt{abcd}}{s}, \quad R = \frac{1}{4} \sqrt{\frac{(ac + bd)(ad + bc)(ab + cd)}{abcd}}.$$

(c) Prove that the area of $ABCD$ is $\sqrt{abcd} = \frac{1}{2} \sqrt{p^2q^2 - (ac - bd)^2}$ where $p, q$ are the lengths of the diagonals $AC, BD$ respectively. \(^1\)

\(^1\)Do you recall what Ptolemy’s Theorem states?
(d) Let $O, I$ be the circumcentre and incentre of $ABCD$ respectively. Let $P$ be the intersection of the diagonals $AC$ and $BD$. Prove that $P, O, I$ are collinear.
Concurrency: Ceva and Menelaus Theorem

**Ceva’s Theorem:**
Given a triangle $ABC$ and points $D, E, F$ on segments $BC, CA, AB$ respectively. Then $AD, BE, CF$ are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Proof: Use areas. □

We now generalize Ceva’s Theorem to a hybrid of Ceva and Menelaus Theorem.

**Ceva and Menelaus’ Theorem**
Given triangle $ABC$ and points $D, E, F$ on lines $BC, CA, AB$ respectively. Suppose $k$ of the points $D, E, F$ are external of the segment $BC, CA, AB$ respectively. Then

1. $AD, BE, CF$ are concurrent if and only if $k = 0$ or $2$, and

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

2. $D, E, F$ are collinear if and only if $k = 1$ or $3$, and

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

An alternative statement of Ceva and Menelaus Theorem can be stated using signed lengths.

Using Ceva and Menalaus’ Theorem, we can also prove Monge’s Theorem. Given two non-intersecting circles, the **internal similitude** of the two circles is defined to be the intersection of their two common internal tangents. The **external similtude** of the two circles is defined to be the intersection of their two common external tangents.

**Monge’s Theorem:** Given three pairwise non-intersecting circles $\omega_1, \omega_2, \omega_3$ with centres $O_1, O_2, O_3$. Let $P_1$ be a point of similitude of $\omega_2$ and $\omega_3$. Define $P_2$ and $P_3$ analogously.

(a) Suppose exactly $k$ of these similitudes are external where $k \in \{1, 3\}$. Prove that $P_1, P_2, P_3$ are collinear.

(b) Suppose exactly $k$ of these similitudes are external where $k \in \{0, 2\}$. Prove that $O_1P_1, O_2P_2, O_3P_3$ are concurrent.

We also introduce several important theorems related to collinearity and concurrency of points.
Desargue’s Theorem: Given two triangles $ABC, DEF$, prove that $AD, BE, CF$ are concurrent if and only if $AB \cap DE, BC \cap EF, CA \cap FD$ are collinear.

Proof: This can be done using repeated applications of Menelaus Theorem or by using projective coordinates by setting the line passing through $AB \cap DE, BC \cap EF, CA \cap FD$ as the line of infinity. We leave the details to you. □

Pappus’ Theorem: Given two lines $l_1, l_2$, let $A, B, C$ be points on $l_1$ appearing in that order and $D, E, F$ be points on $l_2$ appearing in that order. Then $AE \cap BD, BF \cap CE, CD \cap AF$ are collinear.

Proof: Use projective coordinates by setting the line passing through $AE \cap BD, BF \cap CE$ as the line of infinity. We leave the details to you. □

Pascal’s Theorem: Let $A, B, C, D, E, F$ be vertices of a cyclic hexagon. Prove that $AB \cap DE, BC \cap EF, CD \cap FA$ are collinear.


It is important to also notice the degenerate cases of Pascal’s Theorem. i.e. when two more of $A, B, C, D, E, F$ are equal.

Brianchon’s Theorem: Let $ABCDEF$ be a hexagon that circumscribes a circle. Then $AD, BE, CF$ are concurrent.

Proof: Apply the dual of Pascal’s Theorem. □

Again, note the degenerate cases of Brianchon’s Theorem, where a point of the hexagon can be a point of tangency with its incircle.

Corollary: Let $ABCD$ be a bicentric quadrilateral with incircle $\gamma$, touching $AB, BC, CD, DA$ at $P, Q, R, S$ respectively. Prove that $AC, BD, PR, QS$ are concurrent.

Let $ABC$ be a triangle and a point $P$ on the same plane as the triangle. The pedal triangle of $P$ (with respect to $ABC$) is defined to be the triangle with vertices which are the foot of the perpendicular from $P$ on $BC, CA, AB$.

Simson’s Theorem: Given triangle $ABC$ and a point $P$ in the same plane as $ABC$, the pedal triangle of $P$ is degenerate (i.e. a line) if and only if $P$ is on the circumcircle of $ABC$. This line is called the Simson Line of $ABC$ for $P$.

Proof: Angle chase it. □
Exercises for Tuesday, October 7, 2008 and Tuesday, October 14, 2008

1. Given $ABC$ be a triangle, let $P, Q, R$ be the points where the incircle of $ABC$ touch on $BC, CA, AB$ respectively and $X, Y, Z$ be the points where the excircles opposite $A, B, C$ respectively touch $BC, CA, AB$ respectively.

(a) Prove that $AP, BQ, CR$ are concurrent. This point is called the **Gergonne point** of $\triangle ABC$.

(b) Prove that $AX, BY, CZ$ are concurrent. This point is called the **Nagel point** of $\triangle ABC$.

(c) Let $G, I, N$ be the centroid, incentre and Nagel point respectively of $\triangle ABC$. Prove that $I, G, N$ are collinear and \[ \frac{IG}{GN} = \frac{1}{2}. \]

2. Given triangle $ABC$, let lines passing through $B, C$ respectively tangent to the circumcircle of $ABC$ meet at $P$.

(a) Prove that $AP$ is a symmedian of $A$. i.e. the reflection of the median from $A$ about the internal angle bisector of $A$.

(b) Prove that the three symmedians are concurrent. This point of concurrency is called the **Lemoine point** of $\triangle DEF$.

(c) Let $L$ be the Lemoine point of $ABC$ and $D, E, F$ be the feet of the perpendicular on $BC, CA, AB$ from $L$. Prove that $L$ is the centroid of $ABC$.

3. Given three fixed pairwise distinct points $A, B, C$ lying on one straight line in this order. Let $G$ be a circle passing through $A$ and $C$ whose center does not lie on the line $AC$. The tangents to $G$ at $A$ and $C$ intersect each other at a point $P$. The segment $PB$ meets the circle $G$ at $Q$. Show that the point of intersection of the angle bisector of the angle $AQC$ with the line $AC$ does not depend on the choice of the circle $G$.

4. Let $ABC$ be a triangle and $P, Q$ be distinct points on line $BC$ such that $|AP| = |AQ| = s$ where $s$ is the semi-perimeter of $ABC$. Prove that the excircle opposite $A$ is tangent to the circumcircle of $\triangle APQ$.

5. Let $ABC$ be a triangle with circumcircle $\omega$. Let $\Gamma_A$ be the circle tangent to $AB$ and $AC$ and internally tangent to $\omega$, touching $\omega$ at $A'$. Define $B', C'$ analogously. Prove that $AA', BB', CC'$ are concurrent.

6. Given triangle $ABC$, let the incircle $\gamma$ of $ABC$ touch $BC, CA, AB$ at $P, Q, R$ respectively. Let $X$ be any interior point of the $\gamma$ and suppose $PX, QX, RX$ intersect $\gamma$ a second time at $A', B', C'$ respectively. Prove that $AA', BB', CC'$ are concurrent.
7. Given triangle $ABC$ with incircle $\gamma$, let $T_A$ be the point on $\gamma$ such that the circumcircle $\gamma_A$ of $T_ABC$ is tangent to $\gamma$ at $T_A$. Let the common tangent at $T_A$ of $\gamma$ and $\gamma_A$ intersect $BC$ at $P_A$. Define $P_B, P_C, T_B, T_C$ analogously.

(a) Prove that $P_A, P_B, P_C$ are collinear.
(b) Prove that $AT_A, BT_B, CT_C$ are concurrent.

8. Let $ABC$ be a triangle, and $\omega_1, \omega_2, \omega_3$ be circles inside $ABC$ that are tangent (externally) one to each other, such that $\omega_1$ is tangent to $AB$ and $AC$, $\omega_2$ is tangent to $BC$ and $BA$ and $\omega_3$ is tangent to $CA$ and $CB$. Let $D$ be the common point of $\omega_2$ and $\omega_3$, $E$ the common point of $\omega_3$ and $\omega_1$, and $F$ the common point of $\omega_1$ and $\omega_2$. Show that the lines $AD, BE, CF$ are concurrent.

9. Let $ABC$ be a triangle with incentre $I$. Let $D, E, F$ be incentres of $IBC, ICA, IAB$ respectively. Prove that $AD, BE, CF$ are concurrent.

10. Let $ABCD$ be a cyclic quadrilateral, $l_1, l_2$ be lines passing through $D$ perpendicular to $AB$ and $AC$ respectively. Let $T$ be any point on line $AD$ and suppose $BT$ intersects $l_1$ at $X$ and $CT$ intersects $l_2$ at $Y$. Prove that $XY$ passes through a point which is independent of the choice of $T$.

11. Let $ABCD$ be a convex quadrilateral with $BA$ different from $BC$. Denote the incircles of triangle $ABC$ and $ADC$ by $k_1$ and $k_2$ respectively. Suppose that there exists a circle $k$ tangent to ray $BA$ beyond $A$ and to the ray $BC$ beyond $C$, which is also tangent to $AD$ and $CD$.

(a) Prove that $AB + AD = CB + CD$.
(b) Prove that the common external tangent to $k_1$ and $k_2$ intersects on $k$.

12. A point $P$ lies on the side $AB$ of a convex quadrilateral $ABCD$. Let $\omega$ be the incircle of the triangle $CPD$, and let $I$ be its incentre. Suppose that $\omega$ is tangent to the incircles of triangles $APD$ and $BPC$ at points $K$ and $L$, respectively. Let the lines $AC$ and $BD$ meet at $E$, and let the lines $AK$ and $BL$ meet at $F$. Prove that $E, I, F$ are collinear.

13. Let $ABC$ be a triangle and $L$ be a point on side $BC$. Extend rays $AB$ and $AC$ to points $M, N$ respectively such that $\angle ALC = 2\angle AMC$ and $\angle ALB = 2\angle ANB$. Let $O$ be the circumcentre of $\triangle AMN$. Prove that $OL$ is perpendicular to $BC$.

14. Given a triangle $ABC$ with incentre $I$ that touches $BC, CA, AB$ at $D, E, F$ respectively. Let $P$ be the intersection point of the circumcircle of $AEF$ and $ABC$ which is not $A$. Define $Q, R$ analogously. Prove that $PD, QE, RF$ are concurrent.

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2I do not have a solution to this problem yet.